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Multi-mode q-coherent states

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Abstract. We show how to construct a multi-mode operator which satisfies the quantum-Heisenberg–Weyl algebra (H-W$_q$ algebra); we use this operator to create new two-mode realizations of the $SU_q(2)$ and $SU_q(1,1)$ quantum algebras. We also investigate the corresponding coherent states.

1. Introduction

Conventional annihilation (resp. creation) operators for boson modes indexed by $i, j, \ldots$ satisfy

$$[a_i, a_j{}^\dagger] = \delta_{ij} \qquad [a_i, a_j] = 0 \qquad [n_i, a_j] = -\delta_{ij} a_j \tag{1}$$

where we have introduced the number operators $n_i \equiv a_i{}^\dagger a_i$. In a previous paper [1] the authors investigated two-mode states arising from the operator [2]

$$A = a_1 a_2 \{\max(n_1, n_2)\}^{-1/2} \tag{2}$$

which may easily be seen to satisfy the single-mode boson commutation relations

$$[A, A^\dagger] = 1 \qquad [N, A] = -A \tag{3}$$

where we define $N \equiv A^\dagger A = \min(n_1, n_2)$. This may be generalized to three or more modes.

There has been much recent interest in deformations of the conventional boson commutation relations, equation (1). The first such deformation [3] considered was

$$aa^\dagger - qa^\dagger a = 1. \tag{4}$$

The coherent states appropriate to this formulation are related to the classical q-functions of mathematics (see, e.g., the books by Exton [4] or Andrews [5]); we therefore refer to these deformed bosons as M(aths)-bosons. A later version [6, 7] is

$$aa^\dagger - qa^\dagger a = q^{-n} \qquad [n, a^\dagger] = a^\dagger \tag{5}$$

where n is the appropriate number operator. This deformation was a means of obtaining realizations of quantum algebras such as $SU_q(2)$ which arose from physics considerations; we therefore refer to this deformation as a P(hysics)-boson.

We note that both formulations satisfy

$$[a, a^\dagger] \equiv aa^\dagger - a^\dagger a = [n+1]_q - [n]_q \tag{6}$$

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where for M-bosons,

$$[n]_q{}^M \equiv (q^n - 1)/(q - 1)$$

and for P-bosons,

$$[n]_q{}^P \equiv (q^n - q^{-n})/(q - q^{-1}).$$

In general we have

$$[a, (a^\dagger)^r] = (a^\dagger)^{(r-1)}([n + r]_q - [n]_q) \tag{7}$$

and we drop the index M or P as equation (7) holds in either case. We shall usually omit the subscript q on $[n]_q$ when the context makes clear that a value of q is assumed. This result generalizes [8] to

$$[a, F(a^\dagger)]|0\rangle = {}_qD_{a^\dagger} F(a^\dagger)|0\rangle \tag{8}$$

where

$${}_qD_x{}^{\mathrm{M}} F(x) \equiv \frac{F(qx) - F(x)}{x(q-1)}$$

is the q-derivative of classical q-analysis [9], and

$${}_qD_x{}^{\mathrm{P}} F(x) \equiv \frac{F(qx) - F(q^{-1}x)}{x(q - q^{-1})}$$

is the P(hysics) version. We shall find this relation useful in defining the multi-mode q-coherent states. We start with the two-mode case.

2. Multi-mode q-bosons

Consider q-boson modes a_i satisfying either of the commutation relations, equation (4) or equation (5) above, with the appropriate number operators n_i. We shall additionally assume that $[n_i, a_j] = -\delta_{ij} a_j$. Since n_i is a function of a_i and $a_i{}^\dagger$, this ensures that $[n_i, n_j] = 0$.

Now define a two-mode boson operator A by

$$A = a_1 a_2 \{\max([n_1], [n_2])\}^{-1/2} \tag{9}$$

(for both M- and P-bosons). One may verify that with this definition,

$$\begin{aligned} AA^\dagger - qA^\dagger A &= 1 \qquad\qquad [\mathrm{M-case}] \\ AA^\dagger - qA^\dagger A &= q^{-N} \qquad\quad [\mathrm{P-case}] \end{aligned} \tag{10}$$

where the number operator in both cases is given by $N \equiv \min\{n_1, n_2\}$ and satisfies $[N, A] = -A$.

Equation (10) tells us that we have a two-mode realization of both cases of the deformed Heisenberg–Weyl algebra. This in turn enables us to obtain new two-mode realizations of the quantum algebras $SU_q(2)$ and $SU_q(1,1)$ using a q-analogue of the Holstein–Primakoff realization as in [10].

$SU_q(2)$: The Holstein–Primakoff realization of $SU_q(2)$ is given by

$$\begin{aligned} J_+ &= \sqrt{[2\sigma + 1 - N]}A^\dagger \\ J_- &= A\sqrt{[2\sigma + 1 - N]} \\ J_0 &= N - \sigma \end{aligned} \tag{11}$$

where σ is the angular momentum quantum number $\sigma = \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The standard $SU_q(2)$ commutation relations ensue:

$$\begin{aligned} [J_+, J_-] &= [2J_0] \\ \left[J_0, J_\pm\right] &= \pm J_\pm. \end{aligned} \tag{12}$$

$SU_q(1,1)$: In this case the corresponding realization is:

$$\begin{aligned} K_+ &= \sqrt{[2\sigma - 1 + N]}\, A^\dagger \\ K_- &= A\sqrt{[2\sigma - 1 + N]} \\ K_0 &= N + \sigma \end{aligned} \tag{13}$$

where σ is real and positive. These operators satisfy the commutation relations of $SU_q(1,1)$

$$\begin{aligned} [K_+, K_-] &= -[2K_0] \\ \left[K_0, K_\pm\right] &= \pm K_\pm. \end{aligned} \tag{14}$$

Since $[n_1 - n_2, A] = 0$ the two-mode Fock space spanned by the common eigenstates of n_1 and n_2, $\{|i,j\rangle\,;\ i,j = 0,1,2,\cdots\}$, splits into a direct sum of subspaces

$$F_C = \begin{cases} \{|i + C, i\rangle & i = 0, 1, \cdots\} \quad \text{for } C \geq 0 \\ \{|i, i + |C|\rangle & i = 0, 1, \cdots\} \quad \text{for } C < 0 \end{cases}$$

each one of which is invariant under the two-mode boson operators introduced above. In this paper we restrict our attention to the subspace F_0 consisting of diagonal states.

The realization for $SU_q(2)$ given in equation (11) is different from the two-mode realization given in [6, 7]. The realization for $SU_q(1,1)$ generalizes that given in [11] to which it reduces within the diagonal subspace for $\sigma = \frac{1}{2}$.

The above considerations may be readily generalized to the multi-mode case. Define the three-mode q-boson by

$$A = a_1 a_2 a_3 \left\{ \frac{[n_1][n_2][n_3]}{\min([n_1],[n_2],[n_3])} \right\}^{-1/2}. \tag{15}$$

This three-mode q-boson, together with its Hermitian conjugate $A^\dagger$, satisfies the appropriate M or P versions of the deformed Heisenberg–Weyl algebra. This enables three-mode realizations of quantum algebras to be given. If, further, the single-mode operators $a_1, a_2, \ldots$ are replaced by the corresponding multi-boson operators as previously given by the authors [10], one may obtain multi-mode, multi-boson realizations of quantum algebras.

3. Two-mode q-coherent states

We define our two-mode coherent states in the usual way by

$$A|\alpha\rangle = \alpha|\alpha\rangle \tag{16}$$

where A is the two-mode q-boson operator defined in equation (9) above.

From the algebraic discussion of the introduction, we see that a (normalized) solution of equation (16) is given by

$$|\alpha\rangle = \mathcal{N}^{-1} E_q(\alpha A^\dagger)|0,0\rangle \tag{17}$$

where the q-exponential function $E_q(\alpha x)$ satisfies

$$ {}_qD_x E_q(\alpha x) = \alpha E_q(\alpha x) $$

(in both M and P cases). The q-exponential is defined in both cases by

$$ E_q(x) = \sum_{r=0}^{\infty} \frac{x^r}{[r]_q!}. \tag{18} $$

The symbol $[r]_q!$ is defined by $[r]_q! = [r]_q[r-1]_q[r-2]_q \cdots [1]_q$. The normalization constant $\mathcal{N}$ is given by $\mathcal{N}^2 = E_q(|\alpha|^2)$. Choosing a normalized basis

$$ |r, r\rangle = \frac{1}{\sqrt{[r]!}}(A^\dagger)^r|0,0\rangle = \frac{(a_1{}^\dagger a_2{}^\dagger)^r}{[r]!}|0,0\rangle \tag{19} $$

we see that the coherent states are given by

$$ |\alpha\rangle = \mathcal{N}^{-1}\sum_{r=0}^{\infty}\frac{\alpha^r A^{\dagger r}}{[r]!}|0,0\rangle = \mathcal{N}^{-1}\sum_{r=0}^{\infty}\frac{\alpha^r}{\sqrt{[r]!}}|r,r\rangle. \tag{20} $$

It was shown in the conventional case [1] that two-mode coherent states exhibit squeezing; that is, one of the components of the associated electromagnetic field can have dispersion less than the vacuum value of 1/2 (in appropriate units). One may demonstrate that such squeezing effects also occur here. We define the electromagnetic field components in the usual way by $x_1 \equiv (a_1 + a_1{}^\dagger)/2$ and $p_1 \equiv (a_1 - a_1{}^\dagger)/\mathrm{i}\sqrt{2}$; with the corresponding expressions for the second mode. General two-mode components are defined by

$$ X \equiv \lambda x_1 + \mu x_2 = \frac{\lambda}{\sqrt{2}}(a_1 + a_1{}^\dagger) + \frac{\mu}{\sqrt{2}}(a_2 + a_2{}^\dagger) $$

and

$$ P \equiv \lambda p_1 + \mu p_2 = \frac{\lambda}{i\sqrt{2}}(a_1 - a_1{}^\dagger) + \frac{\mu}{\mathrm{i}\sqrt{2}}(a_2 - a_2{}^\dagger). $$

The dispersion of X, for example, is given by

$$ (\Delta X)^2 \equiv \langle X^2\rangle - \langle X\rangle^2. $$

In the coherent state $|\alpha\rangle$ we have $\langle X\rangle = 0 = \langle P\rangle$. Thus, for diagonal states we have

$$ (\Delta X)^2 \equiv \langle X^2\rangle = (\lambda^2+\mu^2)\langle [n_1] + [n_1+1]\rangle + \frac{\lambda\mu}{2}\{\langle a_1a_2 + a_2a_1\rangle + \mathrm{cc}\} \tag{21} $$

and we have not yet made any assumption about the commutator $[a_1, a_2]$. Straightforward calculations give $\langle [n_1]\rangle = |\alpha|^2$, while $\langle [n_1+1]\rangle = q|\alpha|^2 + Q$ where

$$ Q \equiv \begin{cases} 1 & \text{(M-case)} \\ E_q(|\alpha|^2/q)/E_q(|\alpha|^2) & \text{(P-case).} \end{cases} $$

Writing

$$ T \equiv \langle\alpha|a_1a_2|\alpha\rangle = \frac{\alpha}{E_q(|\alpha|^2)}\sum_{r=0}^{\infty}\frac{|\alpha|^{2r}}{[r]_q!}\sqrt{[r+1]_q} $$

and assuming $a_1 a_2 = e^{i\delta} a_2 a_1$ we have for the dispersions

$$(\Delta X)^2 = \frac{\lambda^2 + \mu^2}{2} \{|\alpha|^2(1+q) + Q\} + \lambda\mu(1+\cos\delta)|T|$$
$$(\Delta P)^2 = \frac{\lambda^2 + \mu^2}{2} \{|\alpha|^2(1+q) + Q\} - \lambda\mu(1+\cos\delta)|T|. \qquad (22)$$

We choose $\lambda^2 + \mu^2 = 1$ to maintain the relation between X, P and the corresponding number operator; maximal squeezing occurs for $\delta = 0$ and, just as in the conventional case ($q = 1$), for $\lambda = \mu = 1/\sqrt{2}$.

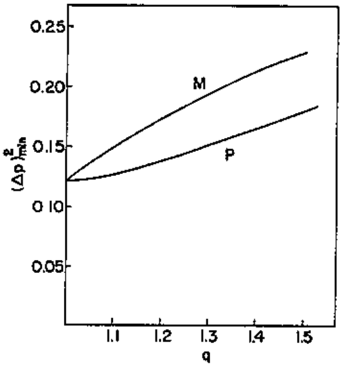

Figure 1. Maximal squeezing for the M- and P-coherent states.

The calculated results, presented in figure 1, show that for P-bosons (symmetric under $q \leftrightarrow 1/q$) the attained squeezing is always less than in the conventional case; this is a feature of these q-boson systems [10]. This is also true for the M-boson case when $q \geq 1$, the region of unrestricted convergence of the infinite series involved.

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